and for a leading edge calculation ($\psi_t = 0$, $\psi_b = 2\pi$)

$$\gamma = \frac{-a_1 \sin 2\pi n + b_1(1 - \cos 2\pi n)}{a_1(1 - \cos 2\pi n) + b_1 \sin 2\pi n} \qquad \text{(D10)}$$

The flow variable vector **P** now may be obtained from the above equations. A point

$$x = k(\phi) \cos\phi \qquad y = k(\phi) \sin\phi$$

is chosen, and thus

$$\psi = \arctan\{t \sin\phi/(\cos\phi + s \sin\phi)\}$$

$$r = k\{(\cos\phi + s \sin\phi)^2 + t^2 \sin^2\phi\}^{1/2}$$

The real and imaginary invariants are evaluated and substituted in (D1) for **P**. The variables ζ and v are given directly, and u is obtained from the relation (3). The ratio v/u for the streamline diagrams discussed in Sec. 6 is seen to be independent of k and r and is a function of ϕ only. A similarity flow thus is obtained.

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Refraction Angles for Luminous Sources Within the Atmosphere

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The preliminary tabulation of the altitude variation of atmospheric density computed by the Air Research and Development Command (1959 ARDC model atmosphere) is used to integrate numerically the equation of atmospheric refraction. The refraction angles are obtained as functions of the altitude and apparent zenith angle of a luminous source and agree, at large altitudes and at zenith angles up to 86°, with Bessel's astronomical values (maximum discrepancy being 2 out of 726 sec of arc). This study also determines the height of the atmosphere, for refraction considerations, to be 26 ± 1 miles.

ASTRONOMICAL refraction refers to the refraction of a ray of light as it passes from a celestial object to the eye of the observer. The angle of astronomical refraction is the angular difference between the true direction of the celestial object and its apparent direction (Fig. 1). The cause of this refraction is the altitude variation of the atmospheric density, for it is well known that a ray of light will be bent as it travels between media of different densities if the angle of incidence at the interface has any value other than zero. It is also well known that the angle of astronomical refraction depends upon the zenith angle of the source and the temperature and pressure at the site of the observer, and a large effort has been devoted to the determination of the relationship between these quantities.[2] It will be shown that the most important relationship needed to obtain the theoretical values of the astronomical refraction angles is that between atmospheric density and the height above the earth's surface. The tables of Bessel (3),[3] giving the values of the astronomical refraction angle as functions of the apparent zenith angle of a celestial body and the temperature and pressure at the observer's site, are among the most reliable tables known,[4] since the theoretical values were altered so as to agree with observational results.[5]

It is, of course, apparent that refraction effects will occur for luminous sources that are located within the sensible atmosphere, and, in fact, it is just this subject with which the investigations to be described are concerned. The necessity of knowing the refraction effect for a source within the atmosphere is apparent if optical methods are employed to determine the tracking accuracy of a radar unit. Fig. 2 indicates the systematic angular errors that may be present in a combined radar-optical system. This figure shows that the total radar pointing error angle θ_t is, if boresight and parallax errors are eliminated, equal to the sum of the angle of atmospheric optical refraction θ_R and the angle between the telescope optical axis and the ray entering the optical system θ_M. The angle θ_M is the angle obtained by either a boresight camera or an optical tracker. Consequently, a measurement of θ_M and a knowledge of θ_R allows the radar error angle to be obtained. This last angle is, essentially, composed of two parts: one due to the radar servo errors and the other due to radar atmospheric refraction. This implies that the value of the angle of radar atmospheric refraction is needed only if the error due to the radar servo is desired. If, however, one is interested in the value of the total radar pointing error, then the angle of atmospheric optical refraction is the only refraction angle needed, and it is obvious that an accurate determination of θ_t requires that the refraction angle be determined accurately.

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[1] Member of Technical Staff, Engineering Mechanics Department.

[2] The names of Bessel, Ivory, and Bouger are intimately associated with this subject. Their investigations, particularly those of Bessel, are collected in Refs. 1 and 2.

[3] Numbers in parentheses indicate References at end of paper.

[4] Another well-known table is the Poulkova Table, published in 1870 by Glyden.

[5] The methods whereby the refraction angles are determined experimentally are given by Olmstead (4).

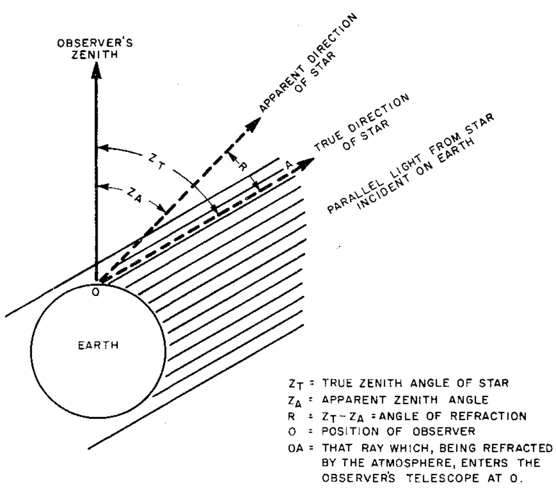

Fig. 1 Refraction by the atmosphere

Approach to the Problem

The derivation of the expression for the angle of astronomical refraction, angle R in Fig. 1, is given in Ref. 1, pp. 205–207. This expression assumes the form

$$R = \int \frac{d\mu}{\mu} \cdot \frac{\sin Z_A}{[(\mu^2 r^2/\mu_1^2 a^2) - \sin^2 Z_A]^{1/2}}$$

where

R = total astronomical refraction angle
Z_A = apparent zenith angle
a = radius of "spherical" earth
μ_1 = index of refraction of the air at the observer
r = radius vector from the center of the earth to that point along the ray where the index of refraction is μ

This is a rigorous equation, depending upon the following assumptions:

1) The atmosphere is formed by an indefinite number of successive strata, each of uniform density. By passing to the limit in which the number of strata becomes infinite and the differences in density in two consecutive strata infinitesimal, one has the case of the continuously varying density of the atmosphere.

2) The successive strata are all concentric with the center of a spherical earth.

It is interesting to note that the upper limit of the refraction equation is identically unity, since the ray commences its bending in space where the density is zero (and hence the index of refraction is unity). Furthermore, it is this refraction equation, integrated into a series solution, which Bessel corrected to account for the observational values of astronomical refraction.

Because the path of any light ray is such that the time taken in travel between any two points will be a minimum (5), one reasons that a luminous source with the sensible atmosphere, viewed at the apparent zenith angle Z_A, must be located on the ray trajectory from a celestial body to the observer, which celestial body is seen at the same angle Z_A. This is illustrated in Fig. 3, where a luminous source at A and the star are both seen at the same apparent zenith angle. This reasoning implies that any method purporting to give the refraction angle as a function of apparent zenith angle and, say, the height of the source above the earth's surface must be capable of predicting the astronomical refraction angle for a particular apparent zenith angle when the altitude of the source becomes infinite. Consequently, it would seem that, if a method of determining the refraction suffered by light from a source within the atmosphere is capable of

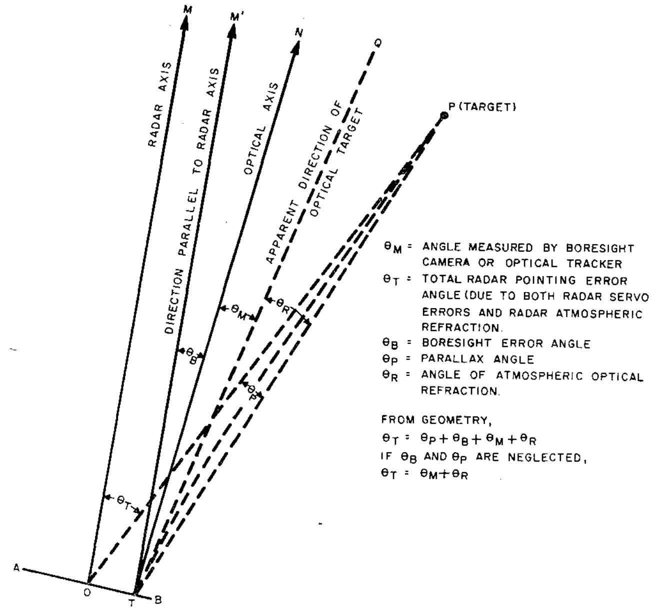

Fig. 2 Geometry of tracking mission systematic errors

predicting the value of astronomical refraction with great accuracy, then the predictions of the refraction when the source is not at an infinite altitude also should be highly accurate.

Instead of assuming an analytical expression for the density (and hence the refractive index) as a function of altitude, as was done by the previously mentioned investigators, it was decided to employ the best atmospheric density vs height data that were currently available from rocket and satellite measurements. To this end, use was made of the 1959 model atmosphere computed by the Air Research and Development Command, Geophysics Research Directorate (6). This atmospheric model defines a smooth density curve that closely fits all of the satellite inferred density data above 240 km and passes through an approximate mean average of the combined satellite and rocket inferred densities below this altitude. This model atmosphere was created by first defining a temperature vs altitude function, which follows approximately the average of observed temperatures up to about 90 or 100 km, the highest altitude for which direct tempera-

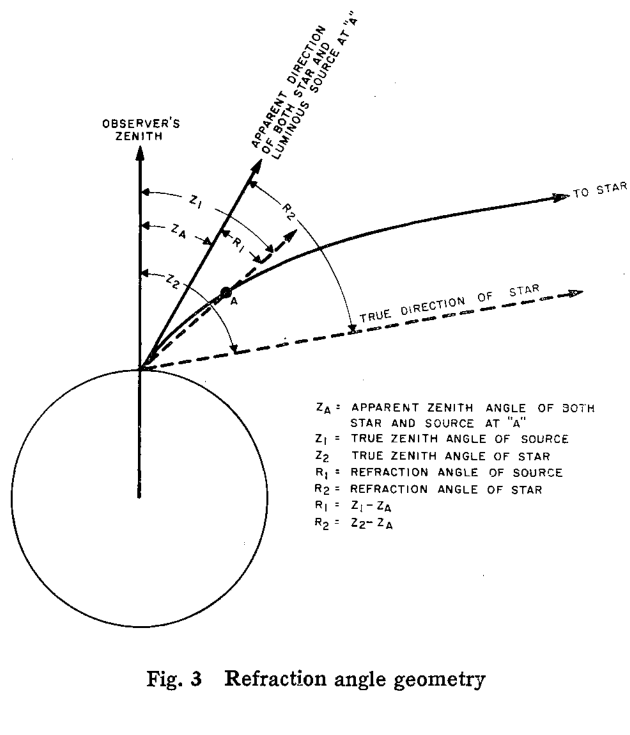

Fig. 3 Refraction angle geometry

ture observations have been made reliably. This temperature vs altitude function, in conjunction with the hydrostatic equation (that equation relating the atmospheric pressure to the altitude, density, and the acceleration of gravity) and the perfect gas law (relating density, pressure, molecular weight, and the atmospheric temperature), gives atmospheric pressures and the densities that agree very well with the average of all measured pressures and densities up to 160 km, the maximum altitude of such observations.

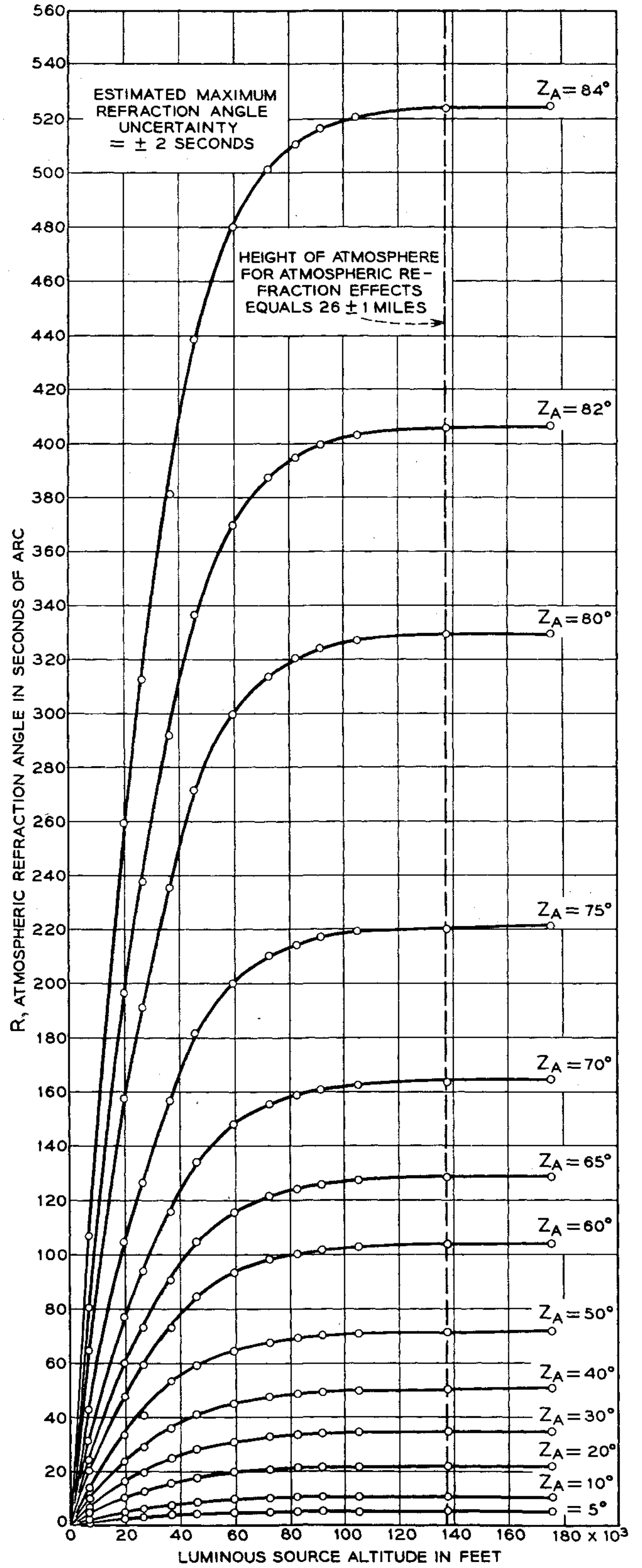

Fig. 4 Refraction angles as a function of source altitude

Table 1 Computed refraction angles and Bessel's values vs apparent zenith angle

Apparent zenith angle Z_A, deg	R_{prog}, sec	R_{Bessel}, sec	Δ, sec
86	726.3	728.2	1.9
84	524.7	526.4	1.7
82	406.6	407.0	0.4
80	330.0	330.1	0.1
75	221.4	221.3	0.1
70	164.2	164.0	0.2
65	128.6	128.5	0.1
60	104.0	103.9	0.1
50	71.7	71.6	0.1
40	50.5	50.5	0.0
30	34.8	34.7	0.1
20	21.9	21.9	0.0
10	10.6	10.6	0.0
5	5.3	5.3	0.0

Employing Sellmeier's optical dispersion formula in the region of negligible absorption (7)

$$\mu^2 - 1 = 2C\rho$$

where

μ = refraction index
ρ = density
C = constant

one has

$$d\mu/\mu = Cd\rho/(1 + 2C\rho)$$

Substituting these two relationships into the equation of atmospheric refraction gives

$$R = C \sin Z_A \times \int_{\rho_0}^{\rho_1} \frac{d\rho}{(1 + 2C\rho)\{[(1 + 2C\rho)r^2/\mu_1^2 a^2] - \sin^2 Z_A\}^{1/2}}$$

where

ρ_0 = density at the observer
ρ_1 = density at the luminous source
μ_1 = refractive index at the observer

It is this equation, together with the 1959 ARDC model atmosphere, which was integrated to give the angle of refraction as a function of the apparent zenith angle and the altitude of the source above the earth's surface ($h = r - a$, where h = altitude, a = radius of earth, and r is the radius vector from the center of the earth to the source). The value of the constant C may be obtained from known values of the refractive index and the density of the air at a particular temperature. For radiation of wavelength 5893 Å,

$$C = 0.12312\ (1/\text{slugs/ft}^3)$$

It is extremely important to point out that the model atmosphere employed is based upon ground level values of temperature and pressure equal to 32.94°F and 29.92 in. Hg, respectively. Consequently, for any apparent zenith angle, the derived curve of refraction angle vs source height above the earth's surface must give, at large heights, a refraction angle that agrees with Bessel's value for the same apparent zenith angle and the same ground level temperature and pressure.

Results (Tabulation and Discussion)

Fig. 4 is a plot of the refraction angles as a function of the height of a luminous source, with apparent zenith angle a parameter. The curve for any apparent zenith angle shows that the refraction angle increases with height and then levels off, asymptotically. The circled points are the actual results of the numerical integration program, a smooth curve being used to connect the individual points. Table 1 lists the apparent zenith angle, the asymptotic value of the refraction angle obtained from the program (R_{prog}) at an altitude of 308,398 ft, and the value of the refraction angle obtained from Bessel's refraction tables (ground level temperature and pressure equal to 32.94°F and 29.92 in. Hg). The difference between these two values is also given. (All refraction angles are in seconds of arc.) The curve for an apparent zenith angle of 86° is not given in Fig. 4 because of scaling difficulties. An enlarged copy of Fig. 4 (11 × 35 in.) may be obtained from the author. The angles for $Z_A = 90°$ were not computed because the integrand of the refraction equation becomes infinite when $r = a$.

Fig. 4 illustrates the following observations:

1) For a source at a particular altitude, the angle of refraction increases as the apparent zenith angle increases.

2) At those apparent zenith angles at which most radar-optical tracking missions take place (i.e., from about 60° to 84°), refraction angles of at least 20 sec of arc (0.1 angular mil) occur when the target is at an altitude of only 7000 ft.

3) It is apparent that all of the curves have approached to within less than 1 sec of arc of their asymptotic values at the height of 26 ± 1 miles. It is this height that is referred to as the "height of the atmosphere for refraction effects."

The extreme accuracy of the program-obtained refraction angles at large altitudes (Table 1) imply that the refraction angles obtained at lesser altitudes are also highly accurate. In fact, since all of the refraction angles listed in Table 1 agree to within at least 1.9 sec of arc with Bessel's values, one can assume that the accuracy of each programmed angle is at least ± 2 sec of arc. Consequently, when the temperature and pressure at the observer are 32.94°F and 29.92 in. Hg, respectively, then the curves given in Fig. 4 can be used to determine the refraction angles to within ±2 sec of arc. This accuracy value is, it is felt, conservative, if apparent zenith angles are restricted to angles less than 84°, since, for these angles, the accuracy of the asymptotic values is equal to or greater than 0.4 sec of arc.

Refraction Angles for Other Temperatures and Pressures

Refraction angles occurring when the ground level temperature and pressure is other than that corresponding to the ARDC model atmosphere may be approximated in the following manner. (Future work will be devoted to a curve-fitting procedure that will give the refraction angles as functions of the apparent zenith angle, source altitude, and the temperature and pressure at the observer. This method will, it is believed, give refraction angles accurate to about ±2 sec of arc instead of the ±20 sec of arc quoted in point 3 below.)

1) The asymptotic value of the refraction angle (i.e., the astronomical value) is obtained from Bessel's tables, corresponding to the apparent zenith angle and to the temperature and pressure at the site of the observer.

2) A smooth curve, connecting the origin of the refraction curve (source height and angle of refraction both equal to zero) and the asymptotic value found in point 1 is drawn so that the curve becomes asymptotic at 26 miles, the "height of the atmosphere for refraction effects."

3) It is estimated by the author that the accuracy of the refraction angles obtained from these curves is better than or equal to ±20 sec of arc.

In this connection, it is important to point out that the ARDC has also created a "tropical" and an "arctic" atmosphere. Although these atmospheres are hydrodynamically consistent (i.e., they represent atmospheres that could occur), nothing is mentioned in Ref. 6 about the agreement between the densities computed for these atmospheres and the experimentally determined densities. Nevertheless, the "tropical" atmosphere density vs height data were used to compute the dependence of the refraction angle upon source height, apparent zenith angle again being a parameter. Twenty-three density vs height points were given (corresponding to an altitude of 100,745 ft), and the results were as follows:

1) The curves (up to 100,745 ft) were, generally, of the same shape as the curves in Fig. 4.

2) The refraction angles (at 100,745 ft) were greater than the asymptotic values obtained from Bessel's tables (using the values of temperature = 89.8°F and pressure = 29.92 in. Hg). It should be remembered that the asymptotic value should be that value occurring at about 26 miles (137,000 ft).

The author concludes that, although the "tropical" atmosphere represents a possible atmosphere, it probably does not give true density vs height data corresponding to the ground level temperature of 89.8°F.

Refs. 8 and 9 are listed to show two of the papers that are concerned with atmospheric refraction.

Acknowledgments

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